

Inverse Trigonometric Functions

Short Answer Type Questions

Q. 1 Find the value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Thinking Process

Use the property, $\tan^{-1}\tan x = x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\cos^{-1}(\cos x) = x$, $x \in [0, \pi]$ to get the answer.

Sol. We know that, $\tan^{-1}\tan x = x$; $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\cos^{-1}\cos x = x$; $x \in [0, \pi]$

$$\begin{aligned}
 \therefore \quad & \tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left(\cos\frac{13\pi}{6}\right) \\
 &= \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{6}\right)\right] + \cos^{-1}\left[\cos\left(\pi + \frac{7\pi}{6}\right)\right] \\
 &= \tan^{-1}\left(-\tan\frac{\pi}{6}\right) + \cos^{-1}\left(-\cos\frac{7\pi}{6}\right) \quad [\because \cos(\pi + \theta) = -\cos \theta] \\
 &= -\tan^{-1}\left(\tan\frac{\pi}{6}\right) + \pi - \left[\cos^{-1}\cos\left(\frac{7\pi}{6}\right)\right] \\
 &\quad \{\because \tan^{-1}(-x) = -\tan^{-1}x; x \in R \text{ and } \cos^{-1}(-x) = \pi - \cos^{-1}x; x \in [-1, 1]\} \\
 &= -\tan^{-1}\left(\tan\frac{\pi}{6}\right) + \pi - \cos^{-1}\left[\cos\left(\pi + \frac{\pi}{6}\right)\right] \\
 &= -\tan^{-1}\left(\tan\frac{\pi}{6}\right) + \pi - \left[\cos^{-1}\left(-\cos\frac{\pi}{6}\right)\right] \quad [\because \cos(\pi + \theta) = -\cos \theta] \\
 &= -\tan^{-1}\left(\tan\frac{\pi}{6}\right) + \pi - \pi + \cos^{-1}\left(\cos\frac{\pi}{6}\right) \quad [\because \cos^{-1}(-x) = \pi - \cos^{-1}x] \\
 &= -\frac{\pi}{6} + 0 + \frac{\pi}{6} = 0
 \end{aligned}$$

Note Remember that, $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) \neq \frac{5\pi}{6}$ and $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) \neq \frac{13\pi}{6}$

Since, $\frac{5\pi}{6} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\frac{13\pi}{6} \notin [0, \pi]$

Q. 2 Evaluate $\cos \left[\cos^{-1} \left(\frac{-\sqrt{3}}{2} \right) + \frac{\pi}{6} \right]$.

Sol. We have, $\cos \left[\cos^{-1} \left(\frac{-\sqrt{3}}{2} \right) + \frac{\pi}{6} \right] = \cos \left[\cos^{-1} \left(\cos \frac{5\pi}{6} \right) + \frac{\pi}{6} \right] \quad \left[\because \cos \frac{5\pi}{6} = \frac{-\sqrt{3}}{2} \right]$
 $= \cos \left(\frac{5\pi}{6} + \frac{\pi}{6} \right) \quad \{ \because \cos^{-1} \cos x = x; \ x \in [0, \pi] \}$
 $= \cos \left(\frac{6\pi}{6} \right)$
 $= \cos(\pi) = -1$

Q. 3 Prove that $\cot \left(\frac{\pi}{4} - 2 \cot^{-1} 3 \right) = 7$.

Sol. We have to prove, $\cot \left(\frac{\pi}{4} - 2 \cot^{-1} 3 \right) = 7$
 $\Rightarrow \left(\frac{\pi}{4} - 2 \cot^{-1} 3 \right) = \cot^{-1} 7$
 $\Rightarrow (2 \cot^{-1} 3) = \frac{\pi}{4} - \cot^{-1} 7$
 $\Rightarrow 2 \tan^{-1} \frac{1}{3} = \frac{\pi}{4} - \tan^{-1} \frac{1}{7}$
 $\Rightarrow 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$
 $\Rightarrow \tan^{-1} \frac{2/3}{1 - (1/3)^2} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$
 $\Rightarrow \tan^{-1} \frac{2/3}{8/9} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$
 $\Rightarrow \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$
 $\Rightarrow \tan^{-1} \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \cdot \frac{1}{7}} = \frac{\pi}{4}$
 $\Rightarrow \tan^{-1} \frac{(21 + 4)/28}{(28 - 3)/28} = \frac{\pi}{4}$
 $\Rightarrow \tan^{-1} \frac{25}{25} = \frac{\pi}{4}$
 $\Rightarrow 1 = \tan \frac{\pi}{4}$
 $\Rightarrow 1 = 1$
 $\Rightarrow \text{LHS} = \text{RHS}$

Hence proved.

Q. 4 Find the value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$.

Sol. We have, $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$

$$= \tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cot^{-1}\left(\cot\frac{\pi}{3}\right) + \tan^{-1}(-1)$$

$$= \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{6}\right)\right] + \cot^{-1}\left[\cot\left(\frac{\pi}{3}\right)\right] + \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{4}\right)\right]$$

$$= \tan^{-1}\left(-\tan\frac{\pi}{6}\right) + \cot^{-1}\left(\cot\frac{\pi}{3}\right) + \tan^{-1}\left(-\tan\frac{\pi}{4}\right)$$

$$\left[\begin{array}{l} \because \tan^{-1}(\tan x) = x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \cot^{-1}(\cot x) = x, x \in (0, \pi) \\ \text{and } \tan^{-1}(-x) = -\tan^{-1}x \end{array} \right]$$

$$= -\frac{\pi}{6} + \frac{\pi}{3} - \frac{\pi}{4} = \frac{-2\pi + 4\pi - 3\pi}{12}$$

$$= \frac{-5\pi + 4\pi}{12} = -\frac{\pi}{12}$$

Q. 5 Find the value of $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$.

Sol. We have, $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}\tan\left(\pi - \frac{\pi}{3}\right)$

$$= \tan^{-1}\left(-\tan\frac{\pi}{3}\right) \quad [\because \tan^{-1}(-x) = -\tan^{-1}x]$$

$$= -\tan^{-1}\tan\frac{\pi}{3} = -\frac{\pi}{3} \quad \left[\because \tan^{-1}(\tan x) = x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right]$$

Note Remember that, $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) \neq \frac{2\pi}{3}$

Since, $\tan^{-1}(\tan x) = x$, if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\frac{2\pi}{3} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Q. 6 Show that $2\tan^{-1}(-3) = \frac{-\pi}{2} + \tan^{-1}\left(\frac{-4}{3}\right)$.

Sol. LHS = $2\tan^{-1}(-3) = -2\tan^{-1}3$ $[\because \tan^{-1}(-x) = -\tan^{-1}x, x \in R]$

$$= -\left[\cos^{-1}\frac{1-3^2}{1+3^2}\right] \quad \left[\because 2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}, x \geq 0 \right]$$

$$= -\left[\cos^{-1}\left(\frac{-8}{10}\right)\right] = -\left[\cos^{-1}\left(\frac{-4}{5}\right)\right]$$

$$= -\left[\pi - \cos^{-1}\left(\frac{4}{5}\right)\right] \quad \{ \because \cos^{-1}(-x) = \pi - \cos^{-1}x, x \in [-1, 1] \}$$

$$= -\pi + \cos^{-1}\left(\frac{4}{5}\right) \quad \left[\text{let } \cos^{-1}\left(\frac{4}{5}\right) = \theta \Rightarrow \cos \theta = \frac{4}{5} \Rightarrow \tan \theta = \frac{3}{4} \Rightarrow \theta = \tan^{-1}\frac{3}{4} \right]$$

$$\begin{aligned}
 &= -\pi + \tan^{-1}\left(\frac{3}{4}\right) = -\pi + \left[\frac{\pi}{2} - \cot^{-1}\left(\frac{3}{4}\right)\right] \\
 &= -\frac{\pi}{2} - \cot^{-1}\frac{3}{4} = -\frac{\pi}{2} - \tan^{-1}\frac{4}{3} \\
 &= -\frac{\pi}{2} + \tan^{-1}\left(\frac{-4}{3}\right) \\
 &= \text{RHS}
 \end{aligned}$$

$$[\because \tan^{-1}(-x) = -\tan^{-1}x]$$

Hence proved.

Q. 7 Find the real solution of

$$\tan^{-1} \sqrt{x(x+1)} + \sin^{-1} \sqrt{x^2 + x + 1} = \frac{\pi}{2}$$

Thinking Process

Convert the $\sin^{-1} \sqrt{x^2 + x + 1}$ into inverse of tangent function and then use the property

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Sol. We have, $\tan^{-1} \sqrt{x(x+1)} + \sin^{-1} \sqrt{x^2 + x + 1} = \frac{\pi}{2}$... (i)

Let $\sin^{-1} \sqrt{x^2 + x + 1} = \theta$

$$\Rightarrow \sin \theta = \frac{\sqrt{x^2 + x + 1}}{1}$$

$$\Rightarrow \tan \theta = \frac{\sqrt{x^2 + x + 1}}{\sqrt{-x^2 - x}}$$

$$\begin{aligned}
 \therefore \theta &= \tan^{-1} \frac{\sqrt{x^2 + x + 1}}{\sqrt{-x^2 - x}} \\
 &= \sin^{-1} \sqrt{x^2 + x + 1}
 \end{aligned}$$

On putting the value of θ in Eq. (i), we get

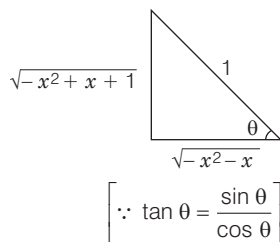
$$\tan^{-1} \sqrt{x(x+1)} + \tan^{-1} \frac{\sqrt{x^2 + x + 1}}{\sqrt{-x^2 - x}} = \frac{\pi}{2}$$

We know that, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$

$$\therefore \tan^{-1} \frac{\sqrt{x(x+1)} + \sqrt{\frac{x^2 + x + 1}{-x^2 - x}}}{1 - \sqrt{x(x+1)} \cdot \sqrt{\frac{x^2 + x + 1}{-x^2 - x}}} = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1} \frac{\sqrt{x^2 + x} + \sqrt{\frac{x^2 + x + 1}{-1(x^2 + x)}}}{1 - \sqrt{(x^2 + x)} \cdot \frac{(x^2 + x + 1)}{-1(x^2 + x)}} = \frac{\pi}{2}$$

$$\Rightarrow \frac{x^2 + x + \sqrt{-(x^2 + x + 1)}}{[1 - \sqrt{-(x^2 + x + 1)}] \sqrt{(x^2 + x)}} = \tan \frac{\pi}{2} = \frac{1}{0}$$



$$\begin{aligned}
\Rightarrow & [1 - \sqrt{-(x^2 + x + 1)}] \sqrt{(x^2 + x)} = 0 \\
\Rightarrow & -(x^2 + x + 1) = 1 \quad \text{or} \quad x^2 + x = 0 \\
\Rightarrow & -x^2 - x - 1 = 1 \quad \text{or} \quad x(x + 1) = 0 \\
\Rightarrow & x^2 + x + 2 = 0 \quad \text{or} \quad x(x + 1) = 0 \\
\therefore & x = \frac{-1 \pm \sqrt{1 - 4 \times 2}}{2} \\
\Rightarrow & x = 0 \quad \text{or} \quad x = -1
\end{aligned}$$

For real solution, we have $x = 0, -1$.

Q. 8 Find the value of $\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos(\tan^{-1}2\sqrt{2})$.

Sol. We have, $\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos(\tan^{-1}2\sqrt{2})$

$$\begin{aligned}
&= \sin\left[\sin^{-1}\left\{\frac{2 \times \frac{1}{3}}{1 + \left(\frac{1}{3}\right)^2}\right\}\right] + \cos\left(\cos^{-1}\frac{1}{3}\right) \quad \left[\because \tan^{-1}x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}\right] \\
&\quad \left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}, -1 \leq x \leq 1 \text{ and } \tan^{-1}(2\sqrt{2}) = \cos^{-1}\frac{1}{3}\right] \\
&= \sin\left[\sin^{-1}\left(\frac{\frac{2}{3}}{1 + \frac{1}{9}}\right)\right] + \frac{1}{3} \quad \{\because \cos(\cos^{-1}x) = x; x \in [-1, 1]\} \\
&= \sin\left[\sin^{-1}\left(\frac{2 \times 9}{3 \times 10}\right)\right] + \frac{1}{3} = \sin\left[\sin^{-1}\left(\frac{3}{5}\right)\right] + \frac{1}{3} \quad [\because \sin(\sin^{-1}x) = x] \\
&= \frac{3}{5} + \frac{1}{3} = \frac{9+5}{15} = \frac{14}{15}
\end{aligned}$$

Q. 9 If $2\tan^{-1}(\cos\theta) = \tan^{-1}(2\operatorname{cosec}\theta)$, then show that $\theta = \frac{\pi}{4}$, where n is any integer.

Thinking Process

Use the property, $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ to prove the desired result.

Sol. We have, $2\tan^{-1}(\cos\theta) = \tan^{-1}(2\operatorname{cosec}\theta)$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos\theta}{1-\cos^2\theta}\right) = \tan^{-1}(2\operatorname{cosec}\theta)$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \left(\frac{2\cos\theta}{\sin^2\theta}\right) = (2\operatorname{cosec}\theta)$$

$$\Rightarrow (\cot\theta \cdot 2\operatorname{cosec}\theta) = (2\operatorname{cosec}\theta) \Rightarrow \cot\theta = 1$$

$$\Rightarrow \cot\theta = \cot\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$$



Q. 10 Show that $\cos\left(2 \tan^{-1} \frac{1}{7}\right) = \sin\left(4 \tan^{-1} \frac{1}{3}\right)$.

Thinking Process

Use the property $2 \tan^{-1} x = \cos^{-1} \frac{1-x^2}{1+x^2}$ and $2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}$, to prove LHS = RHS.

Sol. We have, $\cos\left(2 \tan^{-1} \frac{1}{7}\right) = \sin\left(4 \tan^{-1} \frac{1}{3}\right)$

$$\Rightarrow \cos \left[\cos^{-1} \left(\frac{1 - \left(\frac{1}{7}\right)^2}{1 + \left(\frac{1}{7}\right)^2} \right) \right] = \sin \left[2 \cdot 2 \tan^{-1} \frac{1}{3} \right] \quad \left[\because 2 \tan^{-1} x = \cos^{-1} \left(\frac{1 - x^2}{1 + x^2} \right) \right]$$

$$\Rightarrow \cos \left[\cos^{-1} \left(\frac{\frac{48}{50}}{\frac{49}{49}} \right) \right] = \sin \left[2 \cdot \left(\tan^{-1} \frac{\frac{2}{3}}{1 - \left(\frac{1}{3}\right)^2} \right) \right] \quad \left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1 - x^2} \right) \right]$$

$$\Rightarrow \cos \left[\cos^{-1} \left(\frac{48 \times 49}{50 \times 49} \right) \right] = \sin \left[2 \tan^{-1} \left(\frac{18}{24} \right) \right]$$

$$\Rightarrow \cos \left[\cos^{-1} \left(\frac{24}{25} \right) \right] = \sin \left(2 \tan^{-1} \frac{3}{4} \right)$$

$$\Rightarrow \cos \left[\cos^{-1} \left(\frac{24}{25} \right) \right] = \sin \left(\sin^{-1} \frac{2 \times \frac{3}{4}}{1 + \frac{9}{16}} \right) \quad \left[\because 2 \tan^{-1} x = \sin^{-1} \frac{2x}{1 + x^2} \right]$$

$$\Rightarrow \frac{24}{25} = \sin \left(\sin^{-1} \frac{3/2}{25/16} \right)$$

$$\Rightarrow \frac{24}{25} = \frac{48}{50} \Rightarrow \frac{24}{25} = \frac{24}{25}$$

$$\therefore \text{LHS} = \text{RHS}$$

Hence proved.

Q. 11 Solve the equation $\cos(\tan^{-1} x) = \sin\left(\cot^{-1} \frac{3}{4}\right)$.

Sol. We have, $\cos(\tan^{-1} x) = \sin\left(\cot^{-1} \frac{3}{4}\right)$

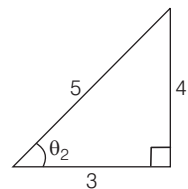
$$\Rightarrow \cos \left(\cos^{-1} \frac{1}{\sqrt{x^2 + 1}} \right) = \sin \left(\sin^{-1} \frac{4}{5} \right)$$

$$\text{Let } \tan^{-1} x = \theta_1 \Rightarrow \tan \theta_1 = \frac{x}{1}$$

$$\Rightarrow \cos \theta_1 = \frac{1}{\sqrt{x^2 + 1}} \Rightarrow \theta_1 = \cos^{-1} \frac{1}{\sqrt{x^2 + 1}}$$

$$\text{and } \cot^{-1} \frac{3}{4} = \theta_2 \Rightarrow \cot \theta_2 = \frac{3}{4}$$

$$\Rightarrow \sin \theta_2 = \frac{4}{5} \Rightarrow \theta_2 = \sin^{-1} \frac{4}{5}$$



$$\Rightarrow \frac{1}{\sqrt{x^2 + 1}} = \frac{4}{5}$$

$$\{\because \cos(\cos^{-1} x) = x, x \in [-1, 1] \text{ and } \sin(\sin^{-1} x) = x, x \in [-1, 1]\}$$

On squaring both sides, we get

$$16(x^2 + 1) = 25$$

$$\Rightarrow 16x^2 = 9$$

$$\Rightarrow x^2 = \left(\frac{3}{4}\right)^2$$

$$\therefore x = \pm \frac{3}{4} = -\frac{3}{4}, \frac{3}{4}$$

Long Answer Type Questions

Q. 12 Prove that $\tan^{-1} \left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \right) = \frac{\pi}{4} + \frac{1}{2} \cos^{-1} x^2$.

Sol. We have,

$$\tan^{-1} \left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \right) = \frac{\pi}{4} + \frac{1}{2} \cos^{-1} x^2$$

$$\therefore \text{LHS} = \tan^{-1} \left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \right) \quad \dots(i)$$

$$[\text{let } x^2 = \cos 2\theta = (\cos^2 \theta - \sin^2 \theta) = 1 - 2\sin^2 \theta = 2\cos^2 \theta - 1]$$

$$\Rightarrow \cos^{-1} x^2 = 2\theta \Rightarrow \theta = \frac{1}{2} \cos^{-1} x^2$$

$$\therefore \sqrt{1+x^2} = \sqrt{1+\cos 2\theta} = \sqrt{1+2\cos^2 \theta - 1} = \sqrt{2} \cos \theta$$

$$\text{and } \sqrt{1-x^2} = \sqrt{1-\cos 2\theta} = \sqrt{1-1+2\sin^2 \theta} = \sqrt{2} \sin \theta$$

$$\therefore \text{LHS} = \tan^{-1} \left(\frac{\sqrt{2} \cos \theta + \sqrt{2} \sin \theta}{\sqrt{2} \cos \theta - \sqrt{2} \sin \theta} \right)$$

$$= \tan^{-1} \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right)$$

$$= \tan^{-1} \left(\frac{1 + \tan \theta}{1 - \tan \theta} \right) = \tan^{-1} \left(\frac{\tan \frac{\pi}{4} + \tan \theta}{1 - \tan \frac{\pi}{4} \cdot \tan \theta} \right)$$

$$= \tan^{-1} \left[\tan \left(\frac{\pi}{4} + \theta \right) \right] \quad \left[\because \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y} \right]$$

$$= \frac{\pi}{4} + \theta = \frac{\pi}{4} + \frac{1}{2} \cos^{-1} x^2$$

$$= \text{RHS}$$

Hence proved.

Q. 13 Find the simplified form of

$$\cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right), \text{ where } x \in \left[\frac{-3\pi}{4}, \frac{\pi}{4}\right].$$

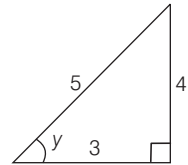
Sol. We have, $\cos^{-1}\left[\frac{3}{5}\cos x + \frac{4}{5}\sin x\right], x \in \left[\frac{-3\pi}{4}, \frac{\pi}{4}\right]$

Let $\cos y = \frac{3}{5}$

$\Rightarrow \sin y = \frac{4}{5}$

$\Rightarrow y = \cos^{-1}\frac{3}{5} = \sin^{-1}\frac{4}{5} = \tan^{-1}\left(\frac{4}{3}\right)$

$\therefore \cos^{-1}[\cos y \cdot \cos x + \sin y \cdot \sin x]$
 $= \cos^{-1}[\cos(y-x)] \quad [\because \cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B]$
 $= y - x = \tan^{-1}\frac{4}{3} - x \quad \left[\because y = \tan^{-1}\frac{4}{3}\right]$



Q. 14 Prove that $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$.

Sol. We have, $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$

$\therefore \text{LHS} = \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$
 $= \tan^{-1}\frac{8}{15} + \tan^{-1}\frac{3}{4}$

Let $\sin^{-1}\frac{8}{17} = \theta_1 \Rightarrow \sin \theta_1 = \frac{8}{17}$

$\Rightarrow \tan \theta_1 = \frac{8}{15} \Rightarrow \theta_1 = \tan^{-1}\frac{8}{15}$

and $\sin^{-1}\frac{3}{5} = \theta_2 \Rightarrow \sin \theta_2 = \frac{3}{5}$

$\Rightarrow \tan \theta_2 = \frac{3}{4} \Rightarrow \theta_2 = \tan^{-1}\frac{3}{4}$

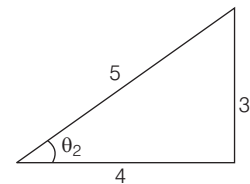
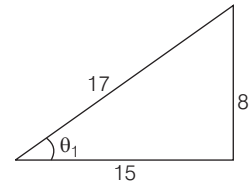
$= \tan^{-1}\left[\frac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{8}{15} \times \frac{3}{4}}\right] \quad \left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= \tan^{-1}\left[\frac{\frac{32+45}{60}}{\frac{60-24}{60}}\right] = \tan^{-1}\left(\frac{77}{36}\right)$

Let $\theta_3 = \tan^{-1}\frac{77}{36} \Rightarrow \tan \theta_3 = \frac{77}{36}$

$\Rightarrow \sin \theta_3 = \frac{77}{\sqrt{5929+1296}} = \frac{77}{85}$

$\therefore \theta_3 = \sin^{-1}\frac{77}{85}$
 $= \sin^{-1}\frac{77}{85} = \text{RHS}$



Hence proved.

Alternate Method

To prove, $\sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{77}{85}$

Let $\sin^{-1} \frac{8}{17} = x$

$\Rightarrow \sin x = \frac{8}{17}$

$\Rightarrow \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \left(\frac{8}{17}\right)^2}$
 $= \sqrt{\frac{289 - 64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}$

Let $\sin^{-1} \frac{3}{5} = y$

$\Rightarrow \sin y = \frac{3}{5} \Rightarrow \sin^2 y = \frac{9}{25}$

$\therefore \cos^2 y = 1 - \frac{9}{25}$

$\Rightarrow \cos^2 y = \left(\frac{4}{5}\right)^2 \Rightarrow \cos y = \frac{4}{5}$

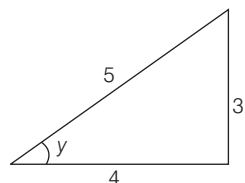
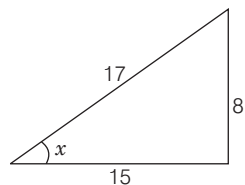
Now, $\sin(x + y) = \sin x \cdot \cos y + \cos x \cdot \sin y$

$$= \frac{8}{17} \cdot \frac{4}{5} + \frac{15}{17} \cdot \frac{3}{5}$$

$$= \frac{32}{85} + \frac{45}{85} = \frac{77}{85}$$

$\Rightarrow (x + y) = \sin^{-1} \left(\frac{77}{85} \right)$

$\Rightarrow \sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{77}{85}$



Q. 15 Show that $\sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5} = \tan^{-1} \frac{63}{16}$.

Sol. We have, $\sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5} = \tan^{-1} \frac{63}{16}$... (i)

Let $\sin^{-1} \frac{5}{13} = x$

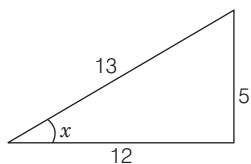
$\Rightarrow \sin x = \frac{5}{13}$

and $\cos^2 x = 1 - \sin^2 x$
 $= 1 - \frac{25}{169} = \frac{144}{169}$

$\Rightarrow \cos x = \sqrt{\frac{144}{169}} = \frac{12}{13}$

$\therefore \tan x = \frac{\sin x}{\cos x} = \frac{5/13}{12/13} = \frac{5}{12}$... (ii)

$\Rightarrow \tan x = 5/12$... (iii)



Again, let

$$\cos^{-1} \frac{3}{5} = y \Rightarrow \cos y = \frac{3}{5}$$

∴

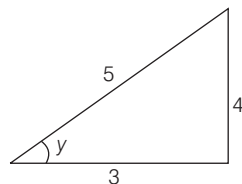
$$\begin{aligned} \sin y &= \sqrt{1 - \cos^2 y} \\ &= \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{1 - \frac{9}{25}} \end{aligned}$$

$$\sin y = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

⇒

$$\tan y = \frac{\sin y}{\cos y} = \frac{4/5}{3/5} = \frac{4}{3}$$

...(iii)



We know that,

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$$

⇒

$$\tan(x + y) = \frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \cdot \frac{4}{3}} \Rightarrow \tan(x + y) = \frac{\frac{15 + 48}{36}}{\frac{36 - 20}{36}}$$

⇒

$$\tan(x + y) = \frac{63/36}{16/36}$$

⇒

$$\tan(x + y) = \frac{63}{16}$$

⇒

$$x + y = \tan^{-1} \frac{63}{16}$$

⇒

$$\tan^{-1} \frac{5}{12} + \tan^{-1} \frac{4}{3} = \tan^{-1} \frac{63}{16}$$

Hence proved.

Q. 16 Prove that $\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{2}{9} = \sin^{-1} \frac{1}{\sqrt{5}}$.

Sol. We have,

$$\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{2}{9} = \sin^{-1} \frac{1}{\sqrt{5}} \quad \dots(i)$$

Let

$$\tan^{-1} \frac{1}{4} = x$$

⇒

$$\tan x = \frac{1}{4}$$

⇒

$$\tan^2 x = \frac{1}{16}$$

⇒

$$\sec^2 x - 1 = \frac{1}{16}$$

⇒

$$\sec^2 x = 1 + \frac{1}{16} = \frac{17}{16}$$

⇒

$$\frac{1}{\cos^2 x} = \frac{17}{16}$$

⇒

$$\cos^2 x = \frac{16}{17}$$

⇒

$$\cos x = \frac{4}{\sqrt{17}}$$

⇒

$$\sin^2 x = 1 - \cos^2 x = 1 - \frac{16}{17} = \frac{1}{17}$$

⇒

$$\sin x = \frac{1}{\sqrt{17}} \quad \dots(ii)$$

Again, let $\tan^{-1} \frac{2}{9} = y$

$$\Rightarrow \tan y = \frac{2}{9} \Rightarrow \tan^2 y = \frac{4}{81}$$

$$\Rightarrow \sec^2 y - 1 = \frac{4}{81}$$

$$\Rightarrow \sec^2 y = \frac{4}{81} + 1 = \frac{85}{81}$$

$$\Rightarrow \cos^2 y = \frac{81}{85} \Rightarrow \cos y = \frac{9}{\sqrt{85}}$$

$$\Rightarrow \sin^2 y = 1 - \cos^2 y = 1 - \frac{81}{85} = \frac{4}{85}$$

$$\Rightarrow \sin y = \frac{2}{\sqrt{85}} \quad \dots (iii)$$

We know that, $\sin(x + y) = \sin x \cdot \cos y + \cos x \cdot \sin y$

$$= \frac{1}{\sqrt{17}} \cdot \frac{9}{\sqrt{85}} + \frac{4}{\sqrt{17}} \cdot \frac{2}{\sqrt{85}}$$

$$= \frac{17}{\sqrt{17} \cdot \sqrt{85}} = \frac{\sqrt{17}}{\sqrt{17} \cdot \sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\Rightarrow (x + y) = \sin^{-1} \frac{1}{\sqrt{5}}$$

$$\Rightarrow \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{2}{9} = \sin^{-1} \frac{1}{\sqrt{5}} \quad \text{Hence proved.}$$

Q. 17 Find the value of $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$.

Thinking Process

Use the properties $2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$ and $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right)$ to get the desired value.

Sol. We have, $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$

$$= 2 \cdot 2 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$$

$$= 2 \cdot \left[\tan^{-1} \frac{\frac{2}{5}}{1 - \left(\frac{1}{5} \right)^2} \right] - \tan^{-1} \frac{1}{239} \quad \left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right]$$

$$= 2 \cdot \left[\tan^{-1} \left(\frac{\frac{2}{5}}{1 - \frac{1}{25}} \right) \right] - \tan^{-1} \frac{1}{239}$$

$$= 2 \cdot \left[\tan^{-1} \left(\frac{2/5}{24/25} \right) \right] - \tan^{-1} \frac{1}{239}$$

$$= 2 \tan^{-1} \frac{5}{12} - \tan^{-1} \frac{1}{239}$$

$$\begin{aligned}
&= \tan^{-1} \frac{2 \cdot \frac{5}{12}}{1 - \left(\frac{5}{12}\right)^2} - \tan^{-1} \frac{1}{239} & \left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right] \\
&= \tan^{-1} \left(\frac{\frac{5}{6}}{1 - \frac{25}{144}} \right) - \tan^{-1} \frac{1}{239} \\
&= \tan^{-1} \left(\frac{144 \times 5}{119 \times 6} \right) - \tan^{-1} \frac{1}{239} \\
&= \tan^{-1} \left(\frac{120}{119} \right) - \tan^{-1} \frac{1}{239} \\
&= \tan^{-1} \left(\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}} \right) & \left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x-y}{1+xy} \right) \right] \\
&= \tan^{-1} \left(\frac{120 \times 239 - 119}{119 \times 239 + 120} \right) \\
&= \tan^{-1} \left[\frac{28680 - 119}{28441 + 120} \right] = \tan^{-1} \frac{28561}{28561} \\
&= \tan^{-1}(1) = \tan^{-1} \left(\tan \frac{\pi}{4} \right) = \frac{\pi}{4}
\end{aligned}$$

Q. 18 Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$ and justify why the other value

$\frac{4+\sqrt{7}}{3}$ is ignored?

Sol. We have, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

$$\therefore \text{LHS} = \tan\left[\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)\right]$$

$$\text{Let } \frac{1}{2}\sin^{-1}\frac{3}{4} = \theta \Rightarrow \sin^{-1}\frac{3}{4} = 2\theta$$

$$\Rightarrow \sin 2\theta = \frac{3}{4} \Rightarrow \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{3}{4}$$

$$\Rightarrow 3 + 3 \tan^2 \theta = 8 \tan \theta$$

$$\Rightarrow 3 \tan^2 \theta - 8 \tan \theta + 3 = 0$$

$$\text{Let } \tan \theta = y$$

$$\therefore 3y^2 - 8y + 3 = 0$$

$$\Rightarrow y = \frac{+8 \pm \sqrt{64 - 4 \times 3 \times 3}}{2 \times 3} = \frac{8 \pm \sqrt{28}}{6}$$

$$= \frac{2[4 \pm \sqrt{7}]}{2 \cdot 3}$$

$$\Rightarrow \tan \theta = \frac{4 \pm \sqrt{7}}{3}$$

$$\Rightarrow \theta = \tan^{-1} \left[\frac{4 + \sqrt{7}}{3} \right]$$

$$\left\{ \text{but } \frac{4 + \sqrt{7}}{3} > \frac{1}{2} \cdot \frac{\pi}{2}, \text{ since } \max \left[\tan \left(\frac{1}{2} \sin^{-1} \frac{3}{4} \right) \right] = 1 \right\}$$

$$\therefore \text{LHS} = \tan \tan^{-1} \left(\frac{4 - \sqrt{7}}{3} \right) = \frac{4 - \sqrt{7}}{3} = \text{RHS}$$

Note Since,

$$-\frac{\pi}{2} \leq \sin^{-1} \frac{3}{4} \leq \pi/2$$

$$\Rightarrow -\frac{\pi}{4} \leq \frac{1}{2} \sin^{-1} \frac{3}{4} \leq \pi/4$$

$$\therefore \tan \left(\frac{-\pi}{4} \right) \leq \tan \frac{1}{2} \left(\sin^{-1} \frac{3}{4} \right) \leq \tan \frac{\pi}{4}$$

$$\Rightarrow -1 \leq \tan \left(\frac{1}{2} \sin^{-1} \frac{3}{4} \right) \leq 1$$

Q. 19 If $a_1, a_2, a_3, \dots, a_n$ is an arithmetic progression with common difference d , then evaluate the following expression.

$$\tan \left[\tan^{-1} \left(\frac{d}{1 + a_1 a_2} \right) + \tan^{-1} \left(\frac{d}{1 + a_2 a_3} \right) + \tan^{-1} \left(\frac{d}{1 + a_3 a_4} \right) \right. \\ \left. + \dots + \tan^{-1} \left(\frac{d}{1 + a_{n-1} a_n} \right) \right]$$

Sol. We have,
and

$$a_1 = a, a_2 = a + d, a_3 = a + 2d \\ d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \dots = a_n - a_{n-1}$$

Given that,

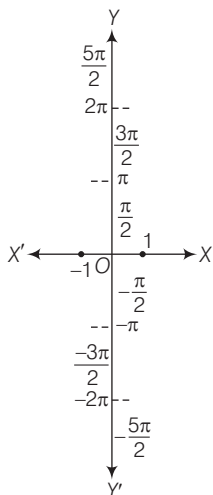
$$\tan \left[\tan^{-1} \left(\frac{d}{1 + a_1 a_2} \right) + \tan^{-1} \left(\frac{d}{1 + a_2 a_3} \right) \right. \\ \left. + \tan^{-1} \left(\frac{d}{1 + a_3 a_4} \right) + \dots + \tan^{-1} \left(\frac{d}{1 + a_{n-1} a_n} \right) \right] \\ = \tan \left[\tan^{-1} \frac{a_2 - a_1}{1 + a_2 \cdot a_1} + \tan^{-1} \frac{a_3 - a_2}{1 + a_3 \cdot a_2} + \dots + \tan^{-1} \frac{a_n - a_{n-1}}{1 + a_n \cdot a_{n-1}} \right] \\ = \tan [(\tan^{-1} a_2 - \tan^{-1} a_1) + (\tan^{-1} a_3 - \tan^{-1} a_2) + \dots + (\tan^{-1} a_n - \tan^{-1} a_{n-1})] \\ = \tan [\tan^{-1} a_n - \tan^{-1} a_1] \\ = \tan \left[\tan^{-1} \frac{a_n - a_1}{1 + a_n \cdot a_1} \right] \quad \left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x - y}{1 + xy} \right) \right] \\ = \frac{a_n - a_1}{1 + a_n \cdot a_1} \quad [\because \tan(\tan^{-1} x) = x]$$

Objective Type Questions

Q. 20 Which of the following is the principal value branch of $\cos^{-1} x$?

- (a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) $(0, \pi)$ (c) $[0, \pi]$ (d) $(0, \pi) - \left\{\frac{\pi}{2}\right\}$

Sol. (c) We know that, the principal value branch of $\cos^{-1} x$ is $[0, \pi]$.



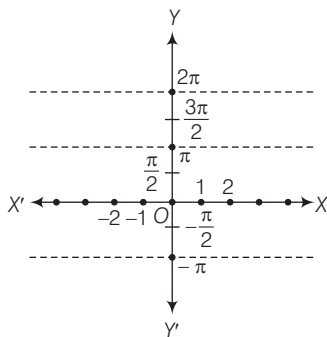
$\therefore$

$$y = \cos^{-1} x$$

Q. 21 Which of the following is the principal value branch of $\operatorname{cosec}^{-1} x$?

- (a) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (b) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ (c) $\left[\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - [0]$

Sol. (d) We know that, the principal value branch of $\operatorname{cosec}^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - 0$.



$\therefore$

$$y = \operatorname{cosec}^{-1} x$$

Q. 22 If $3\tan^{-1}x + \cot^{-1}x = \pi$, then x equals to

- (a) 0 (b) 1 (c) -1 (d) $\frac{1}{2}$

Sol. (b) Given that, $3\tan^{-1}x + \cot^{-1}x = \pi$... (i)

$$\Rightarrow 2\tan^{-1}x + \tan^{-1}x + \cot^{-1}x = \pi$$

$$\Rightarrow 2\tan^{-1}x = \pi - \frac{\pi}{2} \quad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2} \right]$$

$$\Rightarrow 2\tan^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1} \frac{2x}{1-x^2} = \frac{\pi}{2} \quad \left[\because 2\tan^{-1}x = \tan^{-1} \frac{2x}{1-x^2}, \forall x \in (-1, 1) \right]$$

$$\Rightarrow \frac{2x}{1-x^2} = \tan \frac{\pi}{2}$$

$$\Rightarrow \frac{2x}{1-x^2} = \frac{1}{0} \Rightarrow 1-x^2 = 0$$

$$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow x = 1$$

Hence, only $x = 1$ satisfies the given equation.

Note Here, putting $x = -1$ in the given equation, we get

$$3\tan^{-1}(-1) + \cot^{-1}(-1) = \pi$$

$$\Rightarrow 3\tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right] + \cot^{-1}\left[\cot\left(\frac{-\pi}{4}\right)\right] = \pi$$

$$\Rightarrow 3\tan^{-1}\left(-\tan\frac{\pi}{4}\right) + \cot^{-1}\left(-\cot\frac{\pi}{4}\right) = \pi$$

$$\Rightarrow -3\tan^{-1}\left(\tan\frac{\pi}{4}\right) + \pi - \cot^{-1}\left(\cot\frac{\pi}{4}\right) = \pi$$

$$\Rightarrow -3 \cdot \frac{\pi}{4} + \pi - \frac{\pi}{4} = \pi$$

$$\Rightarrow -\pi + \pi = \pi \Rightarrow 0 \neq \pi$$

Hence, $x = -1$ does not satisfy the given equation.

Q. 23 The value of $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right]$ is

- (a) $\frac{3\pi}{5}$ (b) $\frac{-7\pi}{5}$ (c) $\frac{\pi}{10}$ (d) $\frac{-\pi}{10}$

Sol. (d) We have,

$$\sin^{-1}\left(\cos\frac{33\pi}{5}\right) = \sin^{-1}\left[\cos\left(6\pi + \frac{3\pi}{5}\right)\right] = \sin^{-1}\left[\cos\left(\frac{3\pi}{5}\right)\right] \quad [\because \cos(2n\pi + \theta) = \cos \theta]$$

$$= \sin^{-1}\left[\cos\left(\frac{\pi}{2} + \frac{\pi}{10}\right)\right] = \sin^{-1}\left(-\sin\frac{\pi}{10}\right)$$

$$= -\sin^{-1}\left(\sin\frac{\pi}{10}\right) \quad [\because \sin^{-1}(-x) = -\sin^{-1}x]$$

$$= -\frac{\pi}{10} \quad \left[\because \sin^{-1}(\sin x) = x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right]$$

Q. 24 The domain of the function $\cos^{-1}(2x - 1)$ is

(a) $[0, 1]$

(b) $[-1, 1]$

(c) $(-1, 1)$

(d) $[0, \pi]$

Sol. (a) We have, $f(x) = \cos^{-1}(2x - 1)$

$$\begin{aligned} \therefore & -1 \leq 2x - 1 \leq 1 \\ \Rightarrow & 0 \leq 2x \leq 2 \\ \Rightarrow & 0 \leq x \leq 1 \\ \therefore & x \in [0, 1] \end{aligned}$$

Q. 25 The domain of the function defined by $f(x) = \sin^{-1} \sqrt{x - 1}$ is

(a) $[1, 2]$

(b) $[-1, 1]$

(c) $[0, 1]$

(d) None of these

Sol. (a) $\because f(x) = \sin^{-1} \sqrt{x - 1}$

$$\begin{aligned} \Rightarrow & 0 \leq x - 1 \leq 1 & [\because \sqrt{x - 1} \geq 0 \text{ and } -1 \leq \sqrt{x - 1} \leq 1] \\ \Rightarrow & 1 \leq x \leq 2 \\ \therefore & x \in [1, 2] \end{aligned}$$

Q. 26 If $\cos\left(\sin^{-1} \frac{2}{5} + \cos^{-1} x\right) = 0$, then x is equal to

(a) $\frac{1}{5}$

(b) $\frac{2}{5}$

(c) 0

(d) 1

Sol. (b) We have, $\cos\left(\sin^{-1} \frac{2}{5} + \cos^{-1} x\right) = 0$

$$\begin{aligned} \Rightarrow & \sin^{-1} \frac{2}{5} + \cos^{-1} x = \cos^{-1} 0 \\ \Rightarrow & \sin^{-1} \frac{2}{5} + \cos^{-1} x = \cos^{-1} \cos \frac{\pi}{2} \\ \Rightarrow & \sin^{-1} \frac{2}{5} + \cos^{-1} x = \frac{\pi}{2} \\ \Rightarrow & \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} \frac{2}{5} \\ \Rightarrow & \cos^{-1} x = \cos^{-1} \frac{2}{5} & \left[\because \cos^{-1} x + \sin^{-1} x = \frac{\pi}{2} \right] \\ \therefore & x = \frac{2}{5} \end{aligned}$$

Q. 27 The value of $\sin[2 \tan^{-1}(0.75)]$ is

(a) 0.75

(b) 1.5

(c) 0.96

(d) $\sin 1.5$

Sol. (c) We have, $\sin[2 \tan^{-1}(0.75)] = \sin\left(2 \tan^{-1} \frac{3}{4}\right)$ $\left[\because 0.75 = \frac{75}{100} = \frac{3}{4} \right]$

$$\begin{aligned} &= \sin\left[\sin^{-1} \frac{2 \cdot \frac{3}{4}}{1 + \frac{9}{16}}\right] = \sin\left[\sin^{-1} \frac{3/2}{25/16}\right] \\ &= \sin\left[\sin^{-1}\left(\frac{48}{50}\right)\right] = \sin\left[\sin^{-1}\left(\frac{24}{25}\right)\right] = \frac{24}{25} = 0.96 \end{aligned}$$



Q. 28 The value of $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$ is

(a) $\frac{\pi}{2}$

(b) $\frac{3\pi}{2}$

(c) $\frac{5\pi}{2}$

(d) $\frac{7\pi}{2}$

Sol. (a) We have, $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$

$$= \cos^{-1}\cos\left(2\pi - \frac{\pi}{2}\right)$$

$$\left[\because \cos\left(2\pi - \frac{\pi}{2}\right) = \cos\frac{\pi}{2} \right]$$

$$= \cos^{-1}\cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\{\because \cos^{-1}(\cos x) = x, x \in [0, \pi]\}$$

Note Remember that, $\cos^{-1}\left(\cos\frac{3\pi}{2}\right) \neq \frac{3\pi}{2}$

$$\because \frac{3\pi}{2} \notin (0, \pi)$$

Q. 29 The value of $2\sec^{-1}2 + \sin^{-1}\left(\frac{1}{2}\right)$ is

(a) $\frac{\pi}{6}$

(b) $\frac{5\pi}{6}$

(c) $\frac{7\pi}{6}$

(d) 1

Sol. (b) We have, $2\sec^{-1}2 + \sin^{-1}\frac{1}{2} = 2\sec^{-1}\sec\frac{\pi}{3} + \sin^{-1}\sin\frac{\pi}{6}$

$$= 2 \cdot \frac{\pi}{3} + \frac{\pi}{6} \quad [\because \sec^{-1}(\sec x) = x \text{ and } \sin^{-1}(\sin x) = x]$$

$$= \frac{4\pi + \pi}{6} = \frac{5\pi}{6}$$

Q. 30 If $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$, then $\cot^{-1}x + \cot^{-1}y$ equals to

(a) $\frac{\pi}{5}$

(b) $\frac{2\pi}{5}$

(c) $\frac{3\pi}{5}$

(d) π

Sol. (a) We have, $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$

$$\Rightarrow \frac{\pi}{2} - \cot^{-1}x + \frac{\pi}{2} - \cot^{-1}y = \frac{4\pi}{5}$$

$$\Rightarrow -(\cot^{-1}x + \cot^{-1}y) = \frac{4\pi}{5} - \pi$$

$$\left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2} \right]$$

$$\Rightarrow \cot^{-1}x + \cot^{-1}y = -\left(-\frac{\pi}{5}\right)$$

$$\Rightarrow \cot^{-1}x + \cot^{-1}y = \frac{\pi}{5}$$

Q. 31 If $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $a, x \in]0, 1[$,

then the value of x is

- (a) 0 (b) $\frac{a}{2}$ (c) a (d) $\frac{2a}{1-a^2}$

Sol. (d) We have, $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Let $a = \tan \theta \Rightarrow \theta = \tan^{-1} a$

$$\therefore \sin^{-1}\left(\frac{2 \tan \theta}{1 + \tan^2 \theta}\right) + \cos^{-1}\left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}\right) = \tan^{-1} \frac{2x}{1 - x^2}$$

$$\Rightarrow \sin^{-1} \sin 2\theta + \cos^{-1} \cos 2\theta = \tan^{-1} \frac{2x}{1 - x^2}$$

$$\Rightarrow 2\theta + 2\theta = \tan^{-1} \frac{2x}{1 - x^2}$$

$$\Rightarrow 4 \tan^{-1} a = \tan^{-1} \frac{2x}{1 - x^2}$$

$$\Rightarrow 2 \cdot 2 \tan^{-1} a = \tan^{-1} \frac{2x}{1 - x^2}$$

$$\Rightarrow 2 \cdot \tan^{-1} \frac{2a}{1 - a^2} = \tan^{-1} \frac{2x}{1 - x^2} \quad \left[\because 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1 - x^2} \right]$$

$$\Rightarrow \tan^{-1} \frac{2 \cdot \left(\frac{2a}{1 - a^2}\right)}{1 - \left(\frac{2a}{1 - a^2}\right)^2} = \tan^{-1} \left(\frac{2x}{1 - x^2}\right)$$

$$\therefore x = \frac{2a}{1 - a^2}$$

Q. 32 The value of $\cot \left[\cos^{-1} \left(\frac{7}{25} \right) \right]$ is

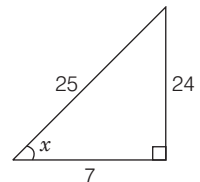
- (a) $\frac{25}{24}$ (b) $\frac{25}{7}$ (c) $\frac{24}{25}$ (d) $\frac{7}{24}$

Sol. (d) We have, $\cot \left[\cos^{-1} \left(\frac{7}{25} \right) \right]$

Let $\cos^{-1} \frac{7}{25} = x$

$$\Rightarrow \cos x = \frac{7}{25}$$

$$\begin{aligned} \therefore \sin x &= \sqrt{1 - \cos^2 x} = \sqrt{1 - \left(\frac{7}{25}\right)^2} \\ &= \sqrt{\frac{625 - 49}{625}} = \frac{24}{25} \end{aligned}$$



$$\begin{aligned}\therefore \cot x &= \frac{\cos x}{\sin x} = \frac{\frac{7}{25}}{\frac{24}{25}} = \frac{7}{24} \quad \dots(i) \\ \Rightarrow x &= \cot^{-1} \left(\frac{7}{24} \right) = \cos^{-1} \left(\frac{7}{25} \right) \\ \therefore \cot \left(\cos^{-1} \frac{7}{25} \right) &= \cot \left(\cot^{-1} \frac{7}{24} \right) = \frac{7}{24} \quad \left[\because \cot^{-1} \frac{7}{24} = \cos^{-1} \frac{7}{25} \right]\end{aligned}$$

Q. 33 The value of $\tan \left(\frac{1}{2} \cos^{-1} \frac{2}{\sqrt{5}} \right)$ is

(a) $2 + \sqrt{5}$

(b) $\sqrt{5} - 2$

(c) $\frac{\sqrt{5} + 2}{2}$

(d) $5 + \sqrt{2}$

Sol. (b) We have,

$$\begin{aligned}\text{Let } \frac{1}{2} \cos^{-1} \frac{2}{\sqrt{5}} &= \theta \\ \Rightarrow \cos^{-1} \frac{2}{\sqrt{5}} &= 2\theta \Rightarrow \cos 2\theta = \frac{2}{\sqrt{5}} \\ \therefore (1 - 2\sin^2 \theta) &= \frac{2}{\sqrt{5}} \\ \Rightarrow 2\sin^2 \theta &= 1 - \frac{2}{\sqrt{5}} \\ \Rightarrow \sin^2 \theta &= \frac{1}{2} - \frac{1}{\sqrt{5}} \\ \Rightarrow \sin \theta &= \sqrt{\frac{1}{2} - \frac{1}{\sqrt{5}}} \\ \therefore \cos^2 \theta &= 1 - \sin^2 \theta \\ &= 1 - \frac{1}{2} + \frac{1}{\sqrt{5}} = \frac{1}{2} + \frac{1}{\sqrt{5}} \\ \Rightarrow \cos \theta &= \sqrt{\frac{1}{2} + \frac{1}{\sqrt{5}}} \\ \therefore \tan \theta &= \sqrt{\frac{\frac{1}{2} - \frac{1}{\sqrt{5}}}{\frac{1}{2} + \frac{1}{\sqrt{5}}}} = \sqrt{\frac{\sqrt{5} - 2}{\sqrt{5} + 2}} \quad \left[\because \tan \theta = \frac{\sin \theta}{\cos \theta} \right] \\ \Rightarrow \theta &= \tan^{-1} \sqrt{\frac{\sqrt{5} - 2}{\sqrt{5} + 2}} = \frac{1}{2} \cos^{-1} \frac{2}{\sqrt{5}} \\ \therefore \tan \left(\frac{1}{2} \cos^{-1} \frac{2}{\sqrt{5}} \right) &= \tan \tan^{-1} \sqrt{\frac{\sqrt{5} - 2}{\sqrt{5} + 2}} \\ &= \sqrt{\frac{\sqrt{5} - 2}{\sqrt{5} + 2} \cdot \frac{\sqrt{5} - 2}{\sqrt{5} - 2}} \\ &= \sqrt{\frac{(\sqrt{5} - 2)^2}{5 - 4}} = \sqrt{5} - 2\end{aligned}$$

Q. 34 If $|x| \leq 1$, then $2\tan^{-1} x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is equal to

- (a) $4\tan^{-1} x$ (b) 0 (c) $\frac{\pi}{2}$ (d) π

Sol. (a) We have, $2\tan^{-1} x + \sin^{-1} \frac{2x}{1+x^2}$

$$\begin{aligned} \text{Let } x &= \tan \theta \\ \therefore 2\tan^{-1} \tan \theta + \sin^{-1} \frac{2 \tan \theta}{1 + \tan^2 \theta} & \quad [\because \tan^{-1}(\tan x) = x] \\ &= 2\theta + \sin^{-1} \sin 2\theta \quad \left[\because \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} \right] \\ &= 2\theta + 2\theta \quad [\because \sin^{-1}(\sin x) = x] \\ &= 4\theta \quad [\because \theta = \tan^{-1} x] \\ &= 4\tan^{-1} x \end{aligned}$$

Q. 35 If $\cos^{-1} \alpha + \cos^{-1} \beta + \cos^{-1} \gamma = 3\pi$, then $\alpha(\beta + \gamma) + \beta(\gamma + \alpha) + \gamma(\alpha + \beta)$ equals to

- (a) 0 (b) 1 (c) 6 (d) 12

Sol. (c) We have, $\cos^{-1} \alpha + \cos^{-1} \beta + \cos^{-1} \gamma = 3\pi$

We know that, $0 \leq \cos^{-1} x \leq \pi$

$$\Rightarrow \cos^{-1} \alpha + \cos^{-1} \beta + \cos^{-1} \gamma = 3\pi$$

If and only if, $\cos^{-1} \alpha = \cos^{-1} \beta = \cos^{-1} \gamma = \pi$

$$\Rightarrow \cos \pi = \alpha = \beta = \gamma$$

$$\Rightarrow -1 = \alpha = \beta = \gamma$$

$$\Rightarrow \alpha = \beta = \gamma = -1$$

$$\begin{aligned} \therefore \alpha(\beta + \gamma) + \beta(\gamma + \alpha) + \gamma(\alpha + \beta) &= -1(-1-1) - 1(-1-1) - 1(-1-1) \\ &= 2 + 2 + 2 = 6 \end{aligned}$$

Q. 36 The number of real solutions of the equation

$$\sqrt{1 + \cos 2x} = \sqrt{2} \cos^{-1}(\cos x) \text{ in } \left[\frac{\pi}{2}, \pi \right] \text{ is}$$

- (a) 0 (b) 1 (c) 2 (d) ∞

Sol. (a) We have, $\sqrt{1 + \cos 2x} = \sqrt{2} \cos^{-1}(\cos x), \left[\frac{\pi}{2}, \pi \right]$

$$\Rightarrow \sqrt{1 + 2\cos^2 x - 1} = \sqrt{2} \cos^{-1}(\cos x)$$

$$\Rightarrow \sqrt{2} \cos x = \sqrt{2} \cos^{-1}(\cos x)$$

$$\Rightarrow \cos x = \cos^{-1}(\cos x)$$

$$\Rightarrow \cos x = x \quad [\because \cos^{-1}(\cos x) = x]$$

which is not true for any real value of x .

Hence, there is no solution possible for the given equation.

Q. 37 If $\cos^{-1} x > \sin^{-1} x$, then

- (a) $\frac{1}{\sqrt{2}} < x \leq 1$ (b) $0 \leq x < \frac{1}{\sqrt{2}}$ (c) $-1 \leq x < \frac{1}{\sqrt{2}}$ (d) $x > 0$

Sol. (c) We have,

$$\cos^{-1} x > \sin^{-1} x, \text{ where } x \in [-1, 1]$$

$\Rightarrow$

$$x < \cos(\sin^{-1} x)$$

$\Rightarrow$

$$x < \cos[\cos^{-1} \sqrt{1-x^2}] \quad \left[\text{let } \sin^{-1} x = \theta \Rightarrow \sin \theta = \frac{x}{1} \right]$$

$$\left[\because \cos \theta = \sqrt{1-\sin^2 \theta} = \sqrt{1-x^2} \Rightarrow \theta = \cos^{-1} \sqrt{1-x^2} \right]$$

$\Rightarrow$

$$x < \sqrt{1-x^2}$$

$\Rightarrow$

$$x^2 < 1-x^2 \Rightarrow 2x^2 < 1$$

$\Rightarrow$

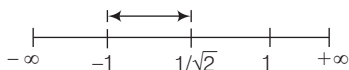
$$x^2 < \frac{1}{2} \Rightarrow x < \pm \left(\frac{1}{\sqrt{2}} \right) \quad \dots(i)$$

Also,

$$-1 \leq x \leq 1 \quad \dots(ii)$$

$\therefore$

$$-1 \leq x \leq \frac{1}{\sqrt{2}}$$



Alternate Method

$$\frac{\pi}{2} - \sin^{-1} x > \sin^{-1} x$$

$$\frac{\pi}{2} > 2\sin^{-1} x \Rightarrow \frac{\pi}{4} > \sin^{-1} x$$

$$\frac{1}{\sqrt{2}} > x \Rightarrow \frac{1}{\sqrt{2}} < x \leq 1$$

We know that,

$$\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

Fillers

Q. 38 The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is

Sol. $\therefore$

$$0 \leq \cos^{-1} x \leq \pi$$

$$\cos^{-1}\left(-\frac{1}{2}\right) = \cos^{-1}\left(\cos \frac{2\pi}{3}\right) = \frac{2\pi}{3}$$

Q. 39 The value of $\sin^{-1}\left(\sin \frac{3\pi}{5}\right)$ is

Sol. $\therefore$

$$-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$$

$$\therefore \sin^{-1}\left(\sin \frac{3\pi}{5}\right) = \sin^{-1}\sin\left(\pi - \frac{2\pi}{5}\right) = \sin^{-1}\left(\sin \frac{2\pi}{5}\right) = \frac{2\pi}{5}$$



Q. 40 If $\cos(\tan^{-1} x + \cot^{-1} \sqrt{3}) = 0$, then the value of x is

Sol. We have, $\cos(\tan^{-1} x + \cot^{-1} \sqrt{3}) = 0$
 $\Rightarrow \tan^{-1} x + \cot^{-1} \sqrt{3} = \cos^{-1} 0$
 $\Rightarrow \tan^{-1} x + \cot^{-1} \sqrt{3} = \cos^{-1} \cos \frac{\pi}{2}$
 $\Rightarrow \tan^{-1} x + \cot^{-1} \sqrt{3} = \frac{\pi}{2}$
 $\Rightarrow \tan^{-1} x = \frac{\pi}{2} - \cot^{-1} \sqrt{3}$
 $\Rightarrow \tan^{-1} x = \tan^{-1} \sqrt{3}$ $\left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \right]$
 $\therefore x = \sqrt{3}$

Q. 41 The set of values of $\sec^{-1} \frac{1}{2}$ is

Sol. Since, domain of $\sec^{-1} x$ is $R - (-1, 1)$.
 $\Rightarrow (-\infty, -1] \cup [1, \infty)$
 So, there is no set of values exist for $\sec^{-1} \frac{1}{2}$.
 So, ϕ is the answer.

Q. 42 The principal value of $\tan^{-1} \sqrt{3}$ is

Sol. $\left[\because \tan^{-1}(\tan x) = x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right]$
 $\tan^{-1} \sqrt{3} = \tan^{-1} \tan \left(\frac{\pi}{3}\right)$
 $= \left(\frac{\pi}{3}\right)$

Q. 43 The value of $\cos^{-1} \left(\cos \frac{14\pi}{3} \right)$ is

Sol. We have, $\cos^{-1} \left(\cos \frac{14\pi}{3} \right) = \cos^{-1} \cos \left(4\pi + \frac{2\pi}{3} \right)$
 $= \cos^{-1} \cos \frac{2\pi}{3}$ $[\because \cos(2n\pi + \theta) = \cos \theta]$
 $= \frac{2\pi}{3}$ $\{ \because \cos^{-1}(\cos x) = x, x \in [0, \pi] \}$
Note Remember that, $\cos^{-1} \left(\cos \frac{14\pi}{3} \right) \neq \frac{14\pi}{3}$
 Since, $\frac{14\pi}{3} \notin [0, \pi]$

Q. 44 The value of $\cos(\sin^{-1} x + \cos^{-1} x)$, where $|x| \leq 1$, is

Sol. $\cos(\sin^{-1} x + \cos^{-1} x)$
 $= \cos \frac{\pi}{2} = 0$ $\left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \right]$

Q. 45 The value of $\tan\left(\frac{\sin^{-1}x + \cos^{-1}x}{2}\right)$, when $x = \frac{\sqrt{3}}{2}$, is

Sol. $\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \right]$ $\tan\left(\frac{\sin^{-1}x + \cos^{-1}x}{2}\right) = \tan\left(\frac{\pi/2}{2}\right)$
 $= \tan\frac{\pi}{4} = 1$

Q. 46 If $y = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then $< y <$

Sol. We have, $y = 2\tan^{-1}x + \sin^{-1}\frac{2x}{1+x^2}$
 $\therefore y = 2\tan^{-1}\tan\theta + \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$ [let $x = \tan\theta$]
 $\Rightarrow y = 2\theta + \sin^{-1}\sin 2\theta$ $\left[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} \right]$
 $\Rightarrow y = 2\theta + 2\theta = 4\theta$ $[\because \theta = \tan^{-1}x]$
 $\Rightarrow y = 4\tan^{-1}x$
 $\therefore -\pi/2 < \tan^{-1}x < \pi/2$
 $\therefore -\frac{4\pi}{2} < 4\tan^{-1}x < 4\pi/2$
 $\Rightarrow -2\pi < 4\tan^{-1}x < 2\pi$
 $\Rightarrow -2\pi < y < 2\pi$ $[\because y = 4\tan^{-1}x]$

Q. 47 The result $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ is true when the value of xy is

Sol. We know that, $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$
 where, $xy > -1$

Q. 48 The value of $\cot^{-1}(-x)$ $x \in R$ in terms of $\cot^{-1}x$ is

Sol. We know that,
 $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in R$

True/False

Q. 49 All trigonometric functions have inverse over their respective domains.

Sol. *False*

We know that, all trigonometric functions have inverse over their restricted domains.

Q. 50 The value of the expression $(\cos^{-1} x)^2$ is equal to $\sec^2 x$.

Sol. False

$$\therefore [\cos^{-1} x]^2 = \left[\sec^{-1} \frac{1}{x} \right]^2 \neq \sec^2 x$$

Q. 51 The domain of trigonometric functions can be restricted to any one of their branch (not necessarily principal value) in order to obtain their inverse functions.

Sol. True

We know that, the domain of trigonometric functions are restricted in their domain to obtain their inverse functions.

Q. 52 The least numerical value, either positive or negative of angle θ is called principal value of the inverse trigonometric function.

Sol. True

We know that, the smallest numerical value, either positive or negative of θ is called the principal value of the function.

Q. 53 The graph of inverse trigonometric function can be obtained from the graph of their corresponding function by interchanging X and Y -axes.

Sol. True

We know that, the graph of an inverse function can be obtained from the corresponding graph of original function as a mirror image (i.e., reflection) along the line $y = x$.

Q. 54 The minimum value of n for which $\tan^{-1} \frac{n}{\pi} > \frac{\pi}{4}$, $n \in N$, is valid is 5.

Sol. False

$$\begin{aligned} \therefore \tan^{-1} \frac{n}{\pi} > \frac{\pi}{4} &\Rightarrow \frac{n}{\pi} > \tan \frac{\pi}{4} \\ \Rightarrow \frac{n}{\pi} &> 1 && \left[\because \tan \frac{\pi}{4} = 1 \right] \\ \Rightarrow n &> \pi \\ \text{So, the minimum value of } n &\text{ is 4.} && [\because n \in N \text{ and } \pi = 3.14...] \end{aligned}$$

Q. 55 The principal value of $\sin^{-1} \left[\cos \left(\sin^{-1} \frac{1}{2} \right) \right]$ is $\frac{\pi}{3}$.

Sol. True

$$\begin{aligned} \text{Given that, } \sin^{-1} \left[\cos \left(\sin^{-1} \frac{1}{2} \right) \right] &= \sin^{-1} \left[\cos \sin^{-1} \left(\sin \frac{\pi}{6} \right) \right] \\ &= \sin^{-1} \left[\cos \frac{\pi}{6} \right] && [\because \sin^{-1} (\sin x) = x] \\ &= \sin^{-1} \frac{\sqrt{3}}{2} \\ &= \sin^{-1} \sin \frac{\pi}{3} = \frac{\pi}{3} \end{aligned}$$

